Critical and noncritical long range entanglement in Klein-Gordon fields

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We investigate the entanglement between two spatially separated intervals in the vacuum state of a free $1D$ Klein-Gordon field by means of explicit computations in the continuum limit of the linear harmonic chain. We demonstrate that the entanglement, which we quantify by the logarithmic negativity, is finite with no further need for renormalization. We find that the quantum correlations are scale-invariant and are determined by a function depending on the ratio of distance to length only. They decay much faster than the classical correlations as in the critical limit long range entanglement decays exponentially for separations larger than the size of the blocks, while classical correlations follow a power law decay. With decreasing distance of the blocks, the entanglement diverges as a power law in the distance. The noncritical regime manifests richer behavior, as the entanglement depends both on the size of the blocks and on their separation. In correspondence with the von Neumann entropy also long-range entanglement distinguishes critical from noncritical systems.

The scaling of block entanglement in both harmonic and spin chains has received considerable attention recently [1]. The scaling of the entanglement entropy $S(\rho_A) = -tr[\rho_A \log \rho_A]$ of a block A has been found to behave in a universal way in one-dimensional critical systems. Explicit computations for the Klein-Gordon massless field using density matrices have shown that the entanglement entropy is proportional to the area of the boundary [2, 3]. Using general conformal field theory methods it has been shown that the entropy of a block with size l scales as $(\bar{c}/3) \log l$ for bosonic fields [4, 5, 6, 7], where $\bar{c} = 1$ is the central charge in the one-dimensional case. This result has been verified analytically [8, 9] using quantum information methods both in critical bosonic spin chains and in critical linear harmonic chains (HC) [10], where the area-law has been proven in higher dimensions in [11, 12]. In noncritical chains however the entropy saturates for blocks larger than the correlation length $\xi \sim m^{-1}$, where m is the energy gap, given in dimensionless units where $\hbar = c = 1$. The von Neumann entropy however, requires further renormalization as it diverges in the continuum limit. Due to this divergence the mutual information $I = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ of two regions A and B has been suggested as a better measure [13] since it admits a finite value in the continuum limit. For critical fields I can still be computed using conformal field theory techniques, where it scales as a power law with the separation between the regions, and decays exponentially in the massive case [14]. However, mutual information is not a genuine measure of entanglement as it includes both classical and quantum correlations as demonstrated by the fact that it does not vanish for separable states [15]. The mutual information is an upper bound to many other entanglement measures, as, for example, the distillable entanglement and the relative entropy of entanglement [15]. Hence we will consider in the following the logarithmic negativity E_{LN} as a quanti-

fier of entanglement [16] for both pure and mixed states, due in part to the relative ease with which it is computed.

We present numerical evidence that the logarithmic negativity admits a finite value in the critical and non critical field limit, making it a promising candidate for studying entanglement of quantum fields. We study the scaling of entanglement between two spatially separated blocks in a free one-dimensional Klein-Gordon field. Moreover a non-zero value of the logarithmic negativity implies distillability [17] in Gaussian systems. At the moment, to the best of our knowledge, there are no methods from conformal field theory, which enable one to compute the logarithmic negativity analytically, and we thus obtain numerically its asymptotic behaviors.

In the following we summarize our main findings before presenting the numerical analysis. Firstly, as there are no length scales in the critical field, any well-defined finite physical property must depend only on a single parameter $r \equiv d/l$, where l is the length of each of the blocks and d is their separation. This property is automatically valid both for the mutual information and the logarithmic negativity both being finite in the continuum limit, due to the following reason. When studying the critical case the continuum limit is taken by taking the lattice constant to zero and the coupling coefficient to infinity in a way that the propagation velocity remains constant. Both the negativity and the mutual information do not depend on the coupling and thus the continuum limit is taken just by increasing the number of oscillators. Thus only the ratio d/l can enter as a parameter in the continuum limit.

Secondly, while the classical correlations between two sites or blocks decay as a power law in the critical regime, we find that quantum correlations measured by E_{LN} decay exponentially with the distance in this regime following $E_{LN} \sim e^{-\beta_c r}$ for $r > 1/2$, where $\beta_c \sim 2\sqrt{2}$ with accuracy better than 1%. This result improves a previ-

ously found lower bound, $E_{LN} \sim e^{-r^2}$ for bosons [18] and fermions [19]. As the blocks approach each other, i.e. $r \to 0$, we find that the E_{LN} diverges as a power law $r^{-\alpha}$, where $\alpha = \frac{1}{3}$ with accuracy better than 1%. Both the mutual information I and the logarithmic negativity E_{LN} are upper bounds to the distillable entanglement. Here we observe that E_{LN} is much tighter though, since I scales as a power law throughout, where we obtained numerically $I \sim r^{-0.05}$. Hence classical correlations exhibit a power law scaling while quantum correlations exhibit an exponential decay.

Thirdly, since the noncritical chain has a length scale proportional to the inverse mass, this system is characterized by two dimensionless parameters $d \to d\xi^{-1}$ and $l \to l\xi^{-1}$, such that $m = 1$, where the critical behavior is obtained in the limit where $d \to 0$ and $l \to 0$. We find that just like the block entropy, long range entanglement allows us to discriminate between critical and noncritical fields. With respect to the blocks' size l, $E_{LN}(l)$ saturates with increasing l, while it diverges for the critical field. Interestingly, the saturation occurs for $l > l_s \sim d + 1$, in contrast to the block entropy for which the saturation occurs for $l > l_s = 1$ ($m = 1$). With respect to the blocks' separation $E_{LN}(d) \sim \exp(-\beta_{nc}(l)\,d^2)$ for $d > l > 1$. As $d \to 0$ the entanglement in both critical and noncritical fields exhibits a similar behaviour since in the noncritical regime (finite l), E_{LN} diverges as a power law as well.

Entanglement between groups of discrete sites has been discussed before in various setups, such as the (discrete) Bose-Hubbard model [20], spin chains [21] and the ion trap [22]. Our work, on the other hand, studies the behavior of long-range entanglement in continuous fields.

Let us start by describing the correspondence between a continuous Klein-Gordon field and the discrete chain, and review the computation of several quantum information measures. The free one-dimensional Klein-Gordon Hamiltonian, $H = \int \mathcal{H}dx$, where

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \qquad (1)$$

corresponds upon discretization with a spacing a to

$$H = \frac{1}{2}\sum_{i=-\infty}^{\infty}\left(a\,\pi_i^2 + \frac{1}{a}(\phi_i - \phi_{i-1})^2 + a\,m^2\phi_i^2\right). \qquad (2)$$

Substituting $\pi_i \to p_i$ and $\phi_i \to q_i$, transforming to circular boundary conditions and writing in dimensionless form we find

$$H = \frac{1}{2}\sum_{n=1}^{N}\left(q_n^2 + p_n^2 - \alpha q_n q_{n+1}\right), \qquad (3)$$

where q_n and p_n are canonical variables ($q_1 = q_{N+1}$), N is the number of oscillators in the chain and $0 < \alpha < 1$ is the coupling constant. The correlation length ξ in units of the oscillators spacing is defined as

$$\xi = \sqrt{\frac{1}{2(1-\alpha)}}. \qquad (4)$$

The continuum limit of the harmonic chain corresponds to Eq. (1) in the strong coupling limit, $\alpha \to 1$, given that $N \to \infty$ and $m = N/\xi$ is kept constant to ensure a constant propagation speed. The system is critical, i.e. $m \to 0$, when $N \ll \xi$.

The spectrum of Eq. (3) is given by

$$\nu_k = \sqrt{1 - \alpha\cos\theta_k}, \qquad (5)$$

where $\theta_k = 2\pi k/N$ and $k = 0, 1, \ldots, N$. Then we can express

$$q_n = \frac{1}{\sqrt{N}}\sum_k\frac{1}{\sqrt{2\nu_k}}[a_k e^{i\theta_k nt} + \text{H.C}],$$
$$p_n = \frac{-i}{\sqrt{N}}\sum_k\frac{\nu_k}{\sqrt{2}}[a_k e^{-i\theta_k nt} - \text{H.C}], \qquad (6)$$

where $[a_k, a_k^\dagger] = 1$, and the two-point vacuum correlation matrices G and H are:

$$G_{ij} = \langle 0|q_i q_j|0\rangle = g_{(i-j)}, H_{ij} = \langle 0|p_i p_j|0\rangle = h_{(i-j)}. \qquad (7)$$

Throughout this paper we consider two separated blocks A and B with the same size $l = \xi L$ and separation $d = \xi D$, where L and D are the number of oscillators in the blocks and their separation, respectively.

For clarity we show how the von Neumann entropy S, the mutual information I and the logarithmic negativity E_{LN} [16] may be computed efficiently for Gaussian states

$$S(A) = \sum_j\left(f(\lambda_j + 1/2) - f(\lambda_j - 1/2)\right), \qquad (8)$$

where $f(x) = x\log x$ and λ_j are the eigenvalues of $iG_A H_A$ with G_A being the restriction of G to a block A. For the logarithmic negativity we find

$$E_{LN} = -\sum_j\log_2\left(\min\{2\tilde{\lambda}_j, 1\}\right), \qquad (9)$$

where $\tilde{\lambda}_j$ are the eigenvalues of $iG_{A\cup B}\tilde{H}_{A\cup B}$, where $\tilde{H}_{A\cup B}$ is obtained from $H_{A\cup B}$ by time-reversal in B, i.e. $p_B \to -p_B$.

In the continuous limit of the critical field the two-point correlation functions $g(x_1 - x_2)$, $h(x_1 - x_2)$ are

$$g(x) \sim g_0 - \log|x| \quad \text{and} \quad h(x) \sim -\frac{1}{x^2}. \qquad (10)$$

The two-point correlation functions in the noncritical case depend on the mass m, where in the asymptotic limit, $x >> m^{-1}$

$$g(x) \sim -\frac{e^{-|x|/m}}{\sqrt{|x|}} \quad \text{and} \quad h(x) \sim \frac{e^{-|x|/m}}{\sqrt{|x|^3}}. \qquad (11)$$

We proceed with the presentation of the numerical results. We examine large chains with $N = 2 \cdot 10^4$ (and $N = 4 \cdot 10^4$ oscillators in order to confirm the continuum

limit). We begin with the critical regime where we take $\alpha = 1 - 10^{-12}$ (deep in the critical limit). In figure 1 we present $\ln E_{LN}$ as a function of r. For $r > 0.5$ the linear approximation practically coincides with the computed values, $E_{LN}(r > 0.5) = E_0 \sim e^{-\beta_c r}$, where the obtained constant is $\beta_c \sim 2\sqrt{2}$ to 1% accuracy.

In the upper inset we observe on a log-log scale the power law correction to the exponential approximation. Assuming $E_{LN} = E_1 \sim r^{-\alpha} e^{-\beta_c r}$, we find $\ln(E_1/E_0) \sim -\alpha \ln r$. For $r < 0.25$ ($\ln r < -1.4$) we obtain numerically $\alpha = \frac{1}{3}$ to 1% accuracy (this number was also numerically observed for critical spin systems in [23]). Note that α is identical to the prefactor of the entanglement entropy $S(l)$ in the critical field.

In the lower inset we confirm that E_{LN} is scale invariant, depending only on d/l in the critical limit. We plot $\ln E_{LN}$ as a function of L, the number of oscillators in each of the blocks, such that the ratio $r \equiv D/L$ is kept constant. The plots are given for different values of r. The curves are approximately constant for L sufficiently large to correspond to the continuum limit. We have also verified the same scale invariance for the mutual information.
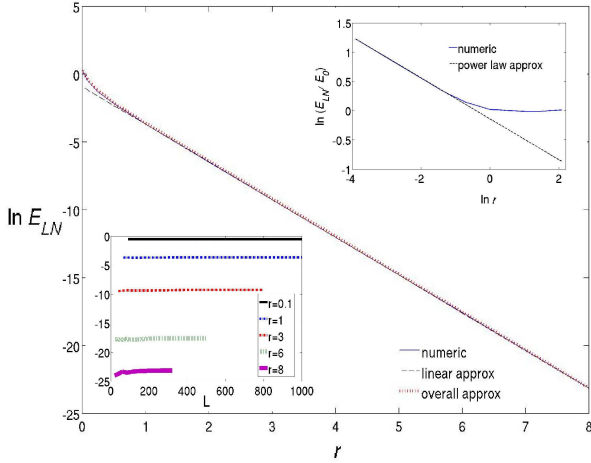

FIG. 1: (Color online). Critical HC: $\ln E_{LN}$ as a function of $r \equiv d/l$. For $r > 0.5$ the linear approximation practically coincides with the computed values, $\beta_c \sim 2\sqrt{2}$. The dotted line is the the overall estimation, (Eq. 12). Upper inset: $\ln(E_1/E_0)$ as a function of $\ln r$. Lower inset: $\ln E_{LN}$ as a function of L for different values of r (D, the number of oscillators that separate the blocks, increases with L).

For arbitrary values of r we find

$$E_{LN}^{\text{critical}} \sim \left(ar^{-\alpha} + f(r) \right) e^{-\beta_c r}, \qquad (12)$$

where $f(r) \sim e^{-\gamma/r}$. Note that as expected $f(r \gg 1) \to 1$ and $f(r \to 0) \to 0$. (Numerically we obtain $\gamma \sim 3/2$ and $a \sim 4/3$.) The dotted line in figure 1 shows Eq. (12) (on logarithmic scale), and provides a very good approximation.

Let us now analyze Eq. (12) with respect to the blocks' size L, keeping their separation D_0 constant. First we note that the first order exponential term $E_0(l) \sim \exp(-\beta_c d_0/l)$ has a saddle point $d^2 E_0/dl^2 = 0$ at $l = \beta_c d_0/2$, in which the scaling turns from exponential at $l \to 0$ to a power of 2. At $l \sim \beta_c d_0$, $E_0(l)$ already scales logarithmically and for $l/d_0 \gg 1$, $E_0(l)$ saturates. However, at this limit the power law correction becomes the dominant factor, where $E_{LN} \sim l^{1/3}$. As the power law is obtained from the slope in a log-log plot, we show in figure 2, $d(\ln E_{LN})/d(\ln L)$ as a function of $\ln L$ for several values of the separations D_0. We also add the saddle points at $L \sim \sqrt{2}D_0$ for each of the curves, which indicate the power of 2. In addition, asymptotically the plots tend to the 1/3 power.
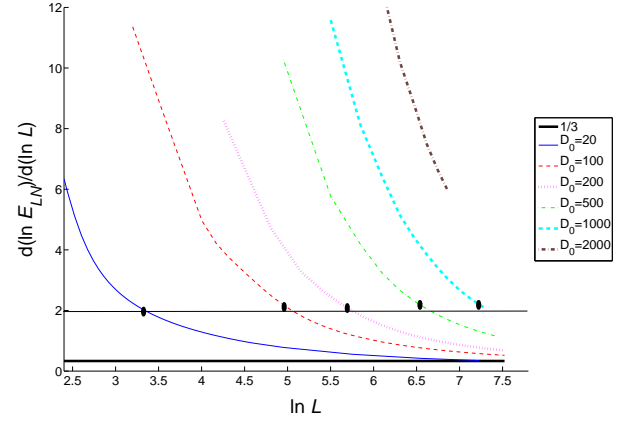

FIG. 2: (Color online). Critical HC: $d(\ln E_{LN})/d(\ln L)$ as a function of L for several values of the separation D_0: $20, 100, 200, 1000, 2000$. The dots indicate the saddle points where the power is ~ 2. $E_{LN}(L/D_0 \to 0) \sim e^L$, where $E_{LN}(L/D_0 \gg 1) \sim L^{1/3}$ (seen for $D_0 = 20$).
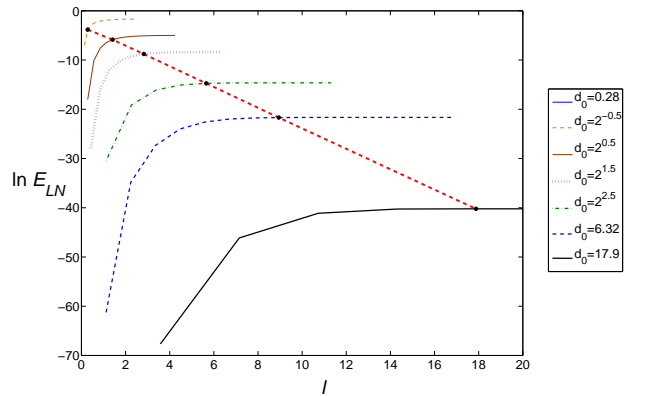

FIG. 3: (Color online). Noncritical chain. $\ln E_{LN}$ as a function of l for various values of d_0. The points $E_{LN}(l = d_0)$ fit a linear curve (broken red) with slope ~ -2.1.

We now turn to investigate long-range entanglement in

the noncritical field, characterized by a length scale m^{-1}, where in units of the particles' spacing in the harmonic chain, the length scale is ξ, defined in Eq. (4). Due to the existence of a length scale, entanglement has to be characterized by two dimensionless parameters d, l given in units of m^{-1}: $d = D/\xi$ and $l = L/\xi$. Numerically we have confirmed that

$$E_{LN}(\xi, D, L) = E_{LN}(\xi\, x, D/x, L/x)$$

in the continuum limit, *i.e.* when we observe no difference in E_{LN} as we simultaneously increase N and α such that N/ξ remains constant. The noncritical regime reduces to the critical one if we take $d \to 0$ and $l \to 0$.

The noncritical regime is characterized by several limits depending on the size of l and d with respect to 1 and with respect to each other. We expect that in correspondence with the von Neumann entropy, in the limit $l >> 1$ the scaling becomes independent of l. Interestingly, we observe that this is indeed true only if also $l \gtrsim d$. In figure 3 we plot $\ln E_{LN}(l)$ for several values of a constant separation d_0. We observe that the entanglement reaches a constant value $E_{LN}^{\text{sat}}(d_0)$, and thus distinguishes noncritical systems from critical ones. The points $E_{LN}(l = d_0)$, which are added for reference, fit a linear curve (broken red line). Saturation occurs at $l > l_s$, where l_s is a linear function of d_0: $l_s \sim 0.75\, d_0 + 1$. For small values of d_0 saturation is obtained for $l_s \sim 1$ ($m = 1$ in our notation). Note that each of the curves starts linearly, showing that the entanglement increases exponentially with l for small values of l.
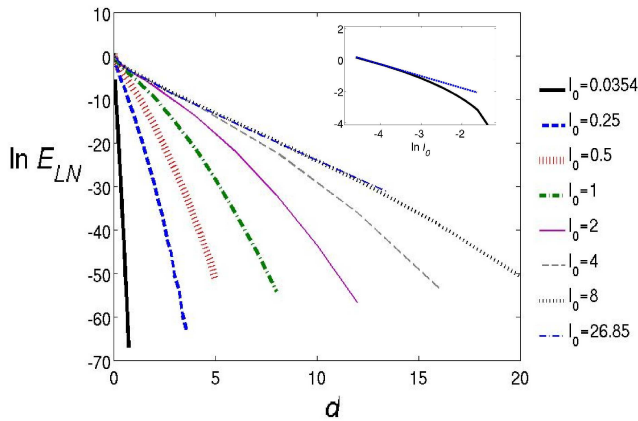

FIG. 4: (Color online). Noncritical chain. $\ln E_{LN}$ as a function of d for various values of l_0. We observe exponential decay with a quadratic function of d (which also depends on l_0). In the intermediate saturation regime $l > d > 1$ the decay is exponentially linear with d. Inset: power law diverge in the $d \to 0$ limit (shown in log-log scale). The power is different than the $1/3$ in the critical limit and in general depends on l_0.

In the opposite limit $d > l$ we observe exponential decay as in the critical regime, but now with different exponent, $E_{LN} \sim \exp\left(-\beta_{nc}(l)d^2\right)$, as can be seen in fig. 4,
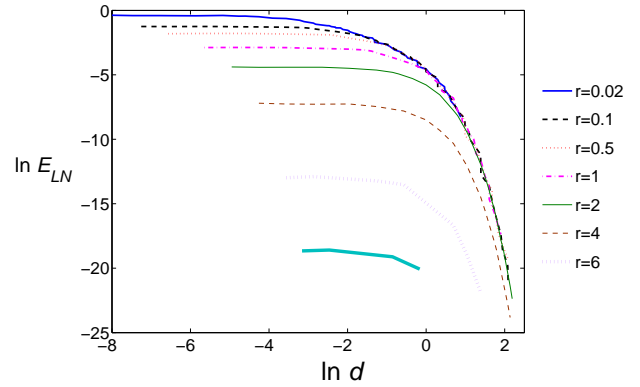

FIG. 5: (Color online). Transition from critical to noncritical chain. $\ln E_{LN}$ as a function of d for constant values of $r = d/l$ (l increases with d). Note that all curves begin with critical behavior as the entanglement is approximately constant. As l becomes close to $m^{-1} = 1$ the noncritical behaviour emerges. All curves with $r < 1$ ($d < l$) coincide in correspondence with the saturation regime, which is independent of l. The curves with $d > l$ do not coincide and characterize the regime where $E_{LN} \sim \exp\left(-\beta_{nc}(l)\, d^2\right)$.

where $E_{LN}(d)$ is shown for several values of the constant blocks size l_0. Note that in the $d \to 0$ limit the entanglement diverges as a power law, as can be seen in the inset in a log-log scale. We obtain that the power is different from the $\alpha = \frac{1}{3}$ in the critical limit and in general depends on l_0. In addition, we observe that for $l_0 \gg d > 1$ the exponential decay becomes linear with d again, corresponding to the intermediate saturation regime.

In order to observe the transition from critical to noncritical behavior we plot in figure 5 $\ln E_{LN}$ as a function of d for constant ratios $r = d/l$. Note that all curves begin with a critical behavior as the entanglement is approximately constant. As l approaches $m^{-1} = 1$ the noncritical behavior emerges and entanglement starts to decay. It can be seen that all curves with $r < 1$ ($d < l$) coincide at a certain point. This corresponds to the saturation regime, in which the entanglement decays as $E_{LN} \sim \exp(-2.25d)$, independent of l. The curves with $d > l$ do not coincide and characterize the regime where $E_{LN} \sim \exp\left(-\beta_{nc}(l)\, d^2\right)$.

We would like to conclude with our main results. The logarithmic negativity, which is a genuine measure of entanglement, is finite in the continuum limit. It is distinguished from the classical correlations especially in the critical limit, where it decays exponentially with the separation, while classical correlations decay as a power law. As the blocks approach each other, the entanglement diverges as a power law, where the power seems to be equal to the universal prefactor of the logarithmic scaling of the von Neumann entropy of a large block. It would be interesting to determine analytically, whether methods from conformal field theory may be applied to the negativity and obtain $\alpha = \bar{c}/3$. We note that much like the en-

tropy of entanglement of a single block, the scaling of long-range entanglement allows us to discriminate critical from noncritical behaviour. Finally, we point out that for the critical field both logarithmic negativity and mutual information are scale invariant and depend only on the ratio between the distance and length of the blocks.

Note added. — When finalizing this paper we became aware of the independent work on long-range entanglement in critical spin-chains drawing similar conclusions

[23].

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